Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$;
- the even values $n = 2$ and $n = 4$.

Hence, copy and complete the following table. [6]

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

(c) Show that $f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(This question continues on the following page)

(Question 1 continued)

(f) Hence, or otherwise, show that $f_n'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n, where $n > 1$ and $a \in \mathbb{R}^+$; [3]

(ii) a point of inflexion with zero gradient for odd values of n, where $n > 1$ and $a \in \mathbb{R}^+$. [2]

Consider the graph of $y = x^n(a - x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a - x)^n = k$ has four solutions for x. [5]

2. [Maximum mark: 24]

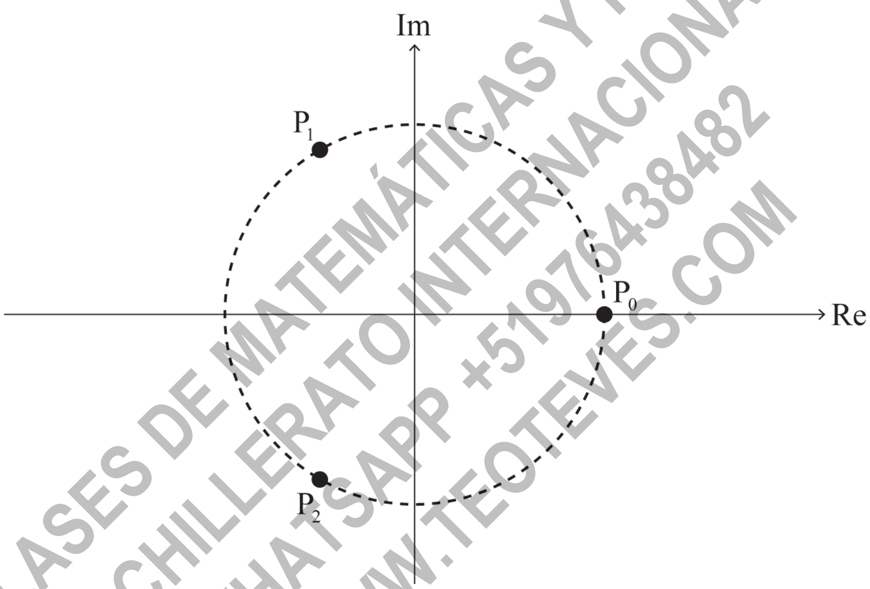
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$. Each root can be represented by a point $P_0, P_1, P_2, \ldots, P_{n-1}$, respectively, on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω. On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1.

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are 1, ω and ω^2. On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre $O(0, 0)$.



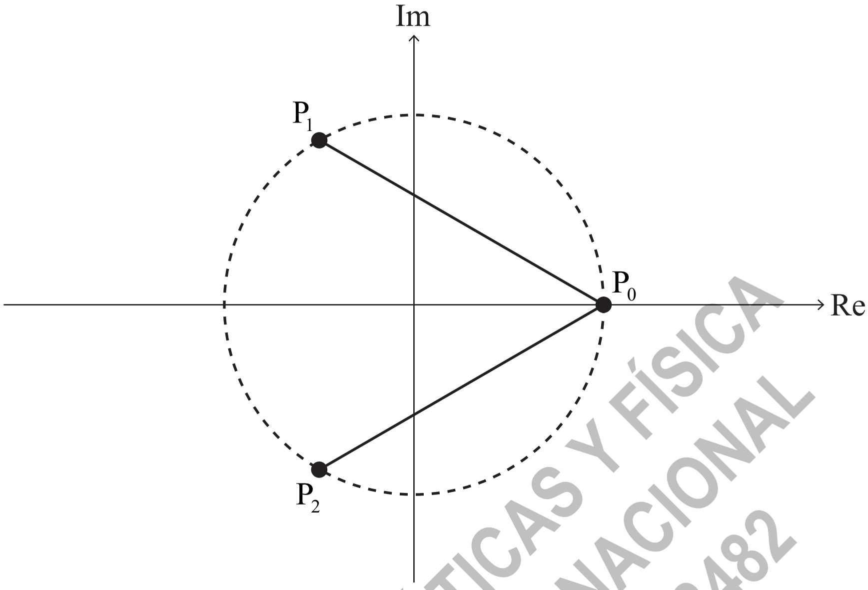
(a) (i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

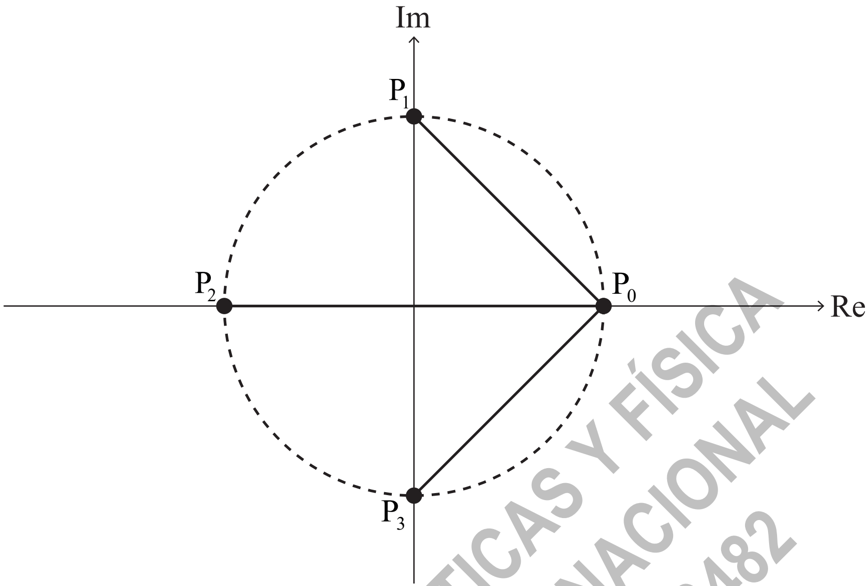
The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2 and ω^3.

(c) By factorizing $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

On the following Argand diagram, the points P_0, P_1, P_2 and P_3 lie on a circle of radius 1 unit with centre $O(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



(d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1, ω, ω^2, ω^3 and ω^4.

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. On an Argand diagram, these roots can be represented by the points $P_0, P_1, P_2, \ldots, P_{n-1}$ respectively where $[P_0P_1], [P_0P_2], \ldots, [P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre $O(0, 0)$.

(e) Suggest a value for $P_0P_1 \times P_0P_2 \times \ldots \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω. [2]

(ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω. [1]

Consider $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + \ldots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

References: