



Mathematics Higher level Paper 3 – sets, relations and groups

Thursday 21 May 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

Consider the set $S_3 = \{p, q, r, s, t, u\}$ of permutations of the elements of the set $\{1, 2, 3\}$, defined by

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, r = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, u = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Let \circ denote composition of permutations, so $a \circ b$ means b followed by a . You may assume that (S_3, \circ) forms a group.

(a) Complete the following Cayley table

\circ	p	q	r	s	t	u
p						
q			t			s
r		u		t	s	q
s		t	u			r
t		s	q	r		
u		r	s	q		

[5]

(b) (i) State the inverse of each element.

(ii) Determine the order of each element.

[6]

(c) Write down the subgroups containing

(i) r ,

(ii) u .

[2]

2. [Maximum mark: 10]

The binary operation $*$ is defined for $x, y \in S = \{0, 1, 2, 3, 4, 5, 6\}$ by

$$x * y = (x^3 y - xy) \pmod{7}.$$

- (a) Find the element e such that $e * y = y$, for all $y \in S$. [2]
- (b) (i) Find the least solution of $x * x = e$.
(ii) Deduce that $(S, *)$ is not a group. [5]
- (c) Determine whether or not e is an identity element. [3]

3. [Maximum mark: 12]

The relation R is defined on \mathbb{Z} by xRy if and only if $x^2 y \equiv y \pmod{6}$.

- (a) Show that the product of three consecutive integers is divisible by 6. [2]
- (b) Hence prove that R is reflexive. [3]
- (c) Find the set of all y for which $5Ry$. [3]
- (d) Find the set of all y for which $3Ry$. [2]
- (e) Using your answers for (c) and (d) show that R is not symmetric. [2]

4. [Maximum mark: 9]

Let X and Y be sets. The functions $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are such that $g \circ f$ is the identity function on X .

- (a) Prove that
- (i) f is an injection,
(ii) g is a surjection. [6]
- (b) Given that $X = \mathbb{R}^+ \cup \{0\}$ and $Y = \mathbb{R}$, choose a suitable pair of functions f and g to show that g is not necessarily a bijection. [3]

5. [Maximum mark: 16]

Consider the sets

$$G = \left\{ \frac{n}{6^i} \mid n \in \mathbb{Z}, i \in \mathbb{N} \right\}, H = \left\{ \frac{m}{3^j} \mid m \in \mathbb{Z}, j \in \mathbb{N} \right\}.$$

(a) Show that $(G, +)$ forms a group where $+$ denotes addition on \mathbb{Q} . Associativity may be assumed. [5]

(b) Assuming that $(H, +)$ forms a group, show that it is a proper subgroup of $(G, +)$. [4]

The mapping $\phi : G \rightarrow G$ is given by $\phi(g) = g + g$, for $g \in G$.

(c) Prove that ϕ is an isomorphism. [7]